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The Chebyshev approximation of a rectangular matrix by matrices of smaller rank as the limit of l_p-approximation

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Abstract: In this paper we investigate a connection between l_p-approximation and the Chebyshev approximation of a rectangular matrix by matrices of smaller rank. We consider also the stationary points of problems (4) and (5) which are connected with these approximations.

Keywords: approximation of matrix, Chebyshev approximation, l_p-approximation, stationary point.

1. Introduction

It is known that the strict Chebyshev solution of an overdetermined system of linear equations is the limit of l_p-solution (see Descloux [2], Fletcher, Grant and Hebden [3], Rice [5, p. 239]). In this paper we investigate the relation between l_p-solution and the Chebyshev solution of an overdetermined nonlinear matrix equation $XY = A$. It is connected with approximation of a rectangular matrix by matrices of smaller rank.

Let M_r be a set of real $(m \times n)$-matrices Z of rank $\leqslant r$. We consider the following problems. For a given natural number r and a give real $(m \times n)$-matrix $A = (a_{ij})$, rank$(A) > r$, find matrix $Z_p \in M_r$ such that

$$\|Z_p - A\|_p = \delta_p \equiv \inf_{Z \in M_r} \|Z - A\|_p, \quad 1 < p < \infty, \tag{1}$$

and find a matrix Z_∞, $Z_\infty \in M_r$, such that

$$\|Z_\infty - A\|_\infty = \delta_\infty \equiv \inf_{Z \in M_r} \|Z - A\|_\infty, \tag{2}$$

where

$$\|A\|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p}, \quad 1 < p < \infty,$$

$$\|A\|_\infty = \max_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n} |a_{ij}|.$$

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The matrix Z_p is an l_p-approximation and Z_∞ is a Chebyshev approximation of A by matrices of rank $\leqslant r$. Since we assume that rank$(A) > r$, we have $\delta_p > 0$ and $\delta_\infty > 0$.

If $Z \in M_r$ then it is possible to express Z in the form $Z = XY$, where X, Y are matrices of dimension $m \times r$, $r \times n$, respectively. This expression is not unique.

Problems (1) and (2) are equivalent to finding an l_p-solution and a Chebyshev solution of the system of mn nonlinear equations with $(m+n)r$ unknowns

$$XY = A. \tag{3}$$

This is connected with the discrete approximation of a function $f(x, y)$ of two variables over a discrete point set

$$S = \left\{ (\xi_i, \eta_j) \colon i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \right\}$$

by sums of products of functions in one variable

$$\sum_{k=1}^{r} g_k(x) h_k(y).$$

Up to now the necessary and sufficient conditions characterizing the l_p-solution and the Chebyshev solution of (3) are unknown. We know only the l_2-solution (see [7]). In [8] we investigated the properties of the stationary points of the problem

$$\min_{X,Y} \| XY - A \|_p, \quad 1 < p < \infty, \tag{4}$$

and we gave an algorithm for computing the stationary points. The properties of the stationary points of the problem

$$\min_{X,Y} \| XY - A \|_\infty \tag{5}$$

were investigated in [7] and [9]. The problems (4) and (5) are equivalent to the problems (1) and (2), respectively.

In this paper we study the relations between the solutions of (1) and (2). We also consider the stationary points of the problems (4) and (5).

In what follows we assume $r < \operatorname{rank}(A)$.

The Chebyshev approximation of A as the limit of l_p-approximation

Let M_r' be a subset of M_r

$$M_r' = \left\{ Z \colon Z \in M_r, \| Z - A \|_p \leqslant \| Z' - A \|_p \right\}, \quad 1 < p \leqslant \infty,$$

where $Z' \in M_r$ and Z' is fixed. The set M_r' is compact. Therefore there exist the solutions of (1) and (2). In [8] we wrote without the proof that the matrix Z_p which was a solution of (1) had rank r. Now we prove it.

Theorem 1. *Let $1 < p < \infty$ and let Z_p be a solution of* (1). *Then $rank(Z_p) = r$.*

Proof. Let $s = \text{rank}(Z_p)$. Suppose a contrario that $s < r$. We define a sequence of matrices W_j ($j = 0, 1, \ldots, n$) in the following way. Let $W_0 = Z_p$. The matrix W_j is formed from the matrix W_{j-1} by replacing of the jth column of W_{j-1} by the jth column of A. Then $W_n = A$, so $\text{rank}(W_n) > r$ and

$$|\text{rank}(W_j) - \text{rank}(W_{j-1})| \leqslant 1, \qquad \|W_j - A\|_p \leqslant \|W_{j-1} - A\|_p.$$

If $\|W_j - A\|_p = \|W_{j-1} - A\|_p$ then $W_j = W_{j-1}$. Therefore there exists k such that $\text{rank}(W_k) = r$ and $\|Z_p - A\|_p > \|W_k - A\|_p$. This is in contradiction with the assumption that Z_p is the solution of (1), which completes the proof. □

The Theorem 1 is not true for $p = \infty$. The following example shows it.

Example 1. Let $m = n = 2$, $r = 1$ and

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Then the matrices

$$Z_\infty^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad Z_\infty^{(2)} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad Z_\infty^{(3)} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

are the solutions of (2) and

$$\text{rank}(Z_\infty^{(1)}) = 0, \qquad \text{rank}(Z_\infty^{(2)}) = \text{rank}(Z_\infty^{(3)}) = 1,$$

so not every solution of (2) has rank r.

Now, we prove that for each matrix A, $\text{rank}(A) > r$, there exists its Chebyshev approximation of rank r.

Theorem 2. *Let* $\text{rank}(A) > r$. *Then there exists a solution of* (2) *which has* rank r.

Proof. Let Z_∞ be a solution of (2) and let $\text{rank}(Z_\infty) = s < r$. Let $W_0 = Z_\infty$. We define a sequence of matrices W_j ($j = 1, \ldots, n$) in the same way as in the proof of Theorem 1. Then $W_n = A$, $\text{rank}(W_n) > r$, and

$$|\text{rank}(W_j) - \text{rank}(W_{j-1})| \leqslant 1, \qquad \|W_j - A\|_\infty \leqslant \|W_{j-1} - A\|_\infty.$$

Therefore there exists k such that $\text{rank}(W_k) = r$ and

$$\|W_k - A\|_\infty \leqslant \|Z_\infty - A\|_\infty.$$

Hence $\|W_k - A\|_\infty = \|Z_\infty - A\|_\infty$ because Z_∞ is the Chebyshev approximation of A by matrices of rank $\leqslant r$. Therefore W_k is the Chebyshev approximation of A and $\text{rank}(W_k) = r$, which completes the proof. □

It is easy to verify that δ_∞ is the limit of δ_p.

Lemma 1. *Let δ_p be given by* (1). *Then*

$$\lim_{p \to \infty} \delta_p = \delta_\infty,$$

where δ_∞ is given in (2).

Proof. For the norms $\|\cdot\|_p$ and $\|\cdot\|_\infty$ we have

$$\|Z\|_\infty \leqslant \|Z\|_p \leqslant (mn)^{1/p}\|Z\|_\infty.$$

Thus, from the definitions of Z_p and Z_∞ it follows that

$$\delta_\infty = \|Z_\infty - A\|_\infty \leqslant \|Z_p - A\|_\infty \leqslant \|Z_p - A\|_p \leqslant \|Z_\infty - A\|_p \leqslant (mn)^{1/p}\delta_\infty,$$

which completes the proof. □

Lemma 2. *Let Z_p be a solution of* (1). *If the sequence $\{Z_p\}$ converges when $p \to \infty$ then*

$$Z_* = \lim_{p\to\infty} Z_p$$

is a solution of (2).

Proof. Suppose that Z_* is not a Chebyshev approximation of A. There exists $\varepsilon > 0$ such that

$$\delta_\infty + \varepsilon \leqslant \|Z_* - A\|_\infty \leqslant \|Z_p - A\|_\infty + \|Z_p - Z_*\|_\infty.$$

If $p \to \infty$ then by Lemma 1 we have $\delta_\infty + \varepsilon \leqslant \delta_\infty$ which is a contradiction. Therefore Z_* is the solution of (2) because rank $(Z_*) \leqslant r$. □

The same argument can be applied to give the following corollary.

Corollary 1. *A cluster point of the sequence of l_p-approximations of A is a Chebyshev approximation.*

The sequence $\{Z_p\}$ is bounded, consequently there exists a cluster point of it. Now we show that the cluster point of rank r satisfies condition RC. We recall the definition of condition RC (see [7]).

Condition RC. Let $Z \in M_r$. We express Z in the form $Z = XY$, where X, Y are matrices of dimensions $m \times r$, $r \times n$, respectively. Matrix Z satisfies condition RC if the columns y_j of matrix Y are the Chebyshev solutions of the linear systems

$$Xy = a_j, \quad j = 1, 2, \ldots, n$$

and the columns x_i of matrix X^T are the Chebyshev solutions of the linear systems

$$Y^T x = d_i, \quad i = 1, 2, \ldots, m$$

where a_j, d_i are the columns of A and A^T, respectively.

Remark. It is easy to verify that the condition RC is not dependent upon the means of expressing Z in the form $Z = XY$, if rank $(Z) = r$.

Theorem 3. *Let Z_* be a cluster point of $\{Z_p\}$. Then Z_* is the solution of* (2) *and if* rank$(Z_*) = r$ *then Z_* satisfies condition* RC.

Proof. The first part of the theorem follows immediately from Corollary 1. Now we prove that Z_* satisfies condition RC. Since Z_* is a cluster point of $\{Z_p\}$, there exists $\{p_k\}$ such that

$$Z_* = \lim_{p_k\to\infty} Z_{p_k}. \tag{6}$$

We omit the index k. Then the notation is simpler. We express Z_* and Z_p in the forms $Z_* = B_* C_*$, $Z_p = B_p C_p$, where B_*, B_p are matrices of dimension $m \times r$. Let $c_j^{(*)}$, $c_j^{(p)}$ denote the jth columns of C_*, C_p, respectively. Suppose that Z_* does not satisfy condition RC. Then either there exists t $(1 \leqslant t \leqslant n)$ such that the tth column of C_* is not the Chebyshev solution of $B_* y = a_t$ or there exists s $(1 \leqslant s \leqslant m)$ such that the sth column of $B^T{}_*$ is not the Chebyshev solution of $C^T{}_* x = d_s$. We assume the first case. So there exists t such that

$$\| B_* c_t^{(*)} - a_t \|_\infty > \inf_{h \in R^r} \| B_* h - a_t \|_\infty . \tag{7}$$

Because $\operatorname{rank}(C_*) = r$ we have

$$\inf_{g \in R^n} \| B_* C_* g - a_t \|_\infty = \inf_{h \in R^r} \| B_* h - a_t \|_\infty .$$

Let

$$u_t(Z) = \| Z e_t - a_t \|_\infty - \inf_{g \in R^n} \| Z g - a_t \|_\infty , \tag{8}$$

where Z denotes an $(m \times n)$-matrix, e_t is the tth column of the identity matrix. From (7) we obtain

$$u_t(Z_*) > 0.$$

Therefore by the lower semi-continuity of the function u_t there exist δ and ε such that

$$u_t(Z) > \varepsilon > 0, \quad Z \in M_\delta ,$$

where

$$M_\delta = \{ Z \colon \| Z - Z_* \|_\infty < \delta \} .$$

From (6) there exists q_0 such that $Z_p \in M_\delta$ for $p > q_0$ and, consequently,

$$u_t(Z_p) > 0 \quad \text{for } p > q_0 . \tag{9}$$

Let

$$\| Z_p g_t^{(p)} - a_t \|_\infty = \inf_g \| Z_p g - a_t \|_\infty .$$

The vector $Z_p g_t^{(p)}$ has the form $B_p d_t^{(p)}$, where $d_t^{(p)} = C_p g_t^{(p)}$. By the properties of the l_p-approximation of A we know that (see Theorem 2.1 in [8])

$$\delta_t^{(p)} \equiv \| B_p c_t^{(p)} - a_t \|_p = \inf_h \| B_p h - a_t \|_p . \tag{10}$$

There exists the limit

$$\delta_t^* = \lim_{p \to \infty} \delta_t^{(p)}$$

because we assume (6). Now, for $p > q_0$ we have (see (8)–(10))

$$\begin{aligned} \delta_t^{(p)} &\geqslant \| B_p c_t^{(p)} - a_t \|_\infty > \varepsilon + \| B_p d_t^{(p)} - a_t \|_\infty \\ &\geqslant \varepsilon + m^{-1/p} \| B_p d_t^{(p)} - a_t \|_p \geqslant \varepsilon + m^{-1/p} \delta_t^{(p)} . \end{aligned} \tag{11}$$

Let $p \to \infty$. Then we obtain

$$\delta_t^* \geqslant \varepsilon + \delta_t^* ,$$

which in turn is a contradiction. Therefore the columns of C_* are the Chebyshev solutions of $B_* y = a_j$ ($j = 1, 2, \ldots, n$). The same argument can be applied to prove that the columns of $B^T{}_*$ are the Chebyshev solutions of $C^T{}_* x = d_i$ ($i = 1, 2, \ldots, m$). So matrix Z_* satisfies condition RC.
□

We conjecture that each cluster point of $\{Z_p\}$ has rank r. Then the assumption in Theorem 3, that $\operatorname{rank}(Z_*) = r$, is not restrictive. It is very difficult to find a matrix for which its l_p-approximation is dependent of p, because we do not know the necessary and sufficient conditions characterizing the solutions of (1). Therefore we can not give any example which would eventually be a counter-example.

3. The relation between the stationary points of (4) and (5)

We recall the definition of the stationary point of problem (4) (see [8]). Let $S_p \in M_r$, $1 < p < \infty$, $S_p = X_p Y_p$, where X_p, Y_p are matrices of dimensions $m \times r$, $r \times n$, respectively. The matrix S_p is a stationary point of (4) if for an $(m \times n)$-matrix $V_p = (v_{ij}^{(p)})$, $V_p \neq 0$, such that

$$v_{ij}^{(p)} = r_{ij}^{(p)} |r_{ij}^{(p)}|^{p-2} \|R_p\|_p^{1-p}, \tag{12}$$

where $R_p = (r_{ij}^{(p)}) = S_p - A$, we have

$$V_p^T X_p = 0, \qquad V_p Y_p^T = 0. \tag{13}$$

However for $p = \infty$ the matrix S_∞, $S_\infty \in M_r$ and $S_\infty = X_\infty Y_\infty$, is a stationary point of (5) if there exists an $(m \times n)$-matrix $V_\infty = (v_{ij}^{(\infty)})$, $V_\infty \neq 0$, such that (see [7,9])

$$V_\infty^T X_\infty = 0, \qquad V_\infty Y_\infty^T = 0 \tag{14}$$

and

$$\begin{aligned} &v_{ij}^{(\infty)} \operatorname{sign}\left(r_{ij}^{(\infty)}\right) \geqslant 0, \quad &(i,j) \in J, \\ &v_{ij}^{(\infty)} = 0, &(i,j) \notin J, \end{aligned} \tag{15}$$

where X_∞ is matrix of dimension $m \times v$ and

$$J \equiv J(R_\infty) = \left\{(i,j) : |r_{ij}^{(\infty)}| = \|R_\infty\|_\infty\right\}, \qquad R_\infty = \left(r_{ij}^{(\infty)}\right) = S_\infty - A.$$

These definitions of the stationary points of (4) and (5) follow from a general definition of the stationary point of an overdetermined system of nonlinear equations (see Osborne and Watson [4], Watson [6]).

Now we consider only the stationary points which have rank r. So we assume

$$\operatorname{rank}(X_p) = \operatorname{rank}(Y_p) = r. \tag{16}$$

This is not restrictive because all the solutions of (4) have rank r. If the assumption (16) holds then the formulas (13) are equivalent to the following ones

$$V_p^T S_p = 0, \qquad V_p S_p^T = 0, \quad 1 < p < \infty, \tag{17}$$

where $S_p = X_p Y_p$. Analogously, if $\operatorname{rank}(X_\infty) = \operatorname{rank}(Y_\infty) = r$ then the formulas (14) are equivalent to the following ones

$$V_\infty^T S_\infty = 0, \qquad V_\infty S_\infty^T = 0, \tag{18}$$

where $S_\infty = X_\infty Y_\infty$.

We consider the following example.

Example 2. Let $m = n = 3$, $r = 2$ and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We consider the following matrices of rank 2

$$G^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad G^{(2)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad G^{(3)} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

For $R^{(k)} = G^{(k)} - A$ the matrices $V_p^{(k)}$ with the elements (12) are equal

$$V_p^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad V_p^{(2)} = \begin{bmatrix} -4^q & 4^q & 0 \\ 4^q & -4^q & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad V_p^{(3)} = \begin{bmatrix} -9^q & -9^q & 9^q \\ -9^q & -9^q & 9^q \\ 9^q & 9^q & -9^q \end{bmatrix},$$

where $q = (1-p)/p$. The matrices $G^{(k)}$ are the stationary points of (4) for each p because the conditions (17) are satisfied for $k = 1, 2, 3$. From the paper [7] it follows that $\delta_2 = 1$, so the matrices $G^{(k)}$ ($k = 1, 2, 3$) are also the l_2-approximations of A. Let

$$V_\infty^{(k)} = \lim_{p\to\infty} V_p^{(k)}, \quad P = 1, 2, 3.$$

It is easy to verify that $G^{(k)}$ and $V_\infty^{(k)}$ satisfy (15) and (18). It means that $G^{(k)}$ ($k = 1, 2, 3$) are the stationary points of (5). Moreover, the matrix $G^{(3)}$ is the Chebyshev approximation of A, because from [8] we obtain that $\frac{1}{3} \leqslant \delta_\infty \leqslant 1$. Thus for $G^{(3)}$ the lower bound is reached.

If the sequence $\{S_p\}$ is bounded then there exists $\{p_k\}$ such that there exists the limit

$$S_* = \lim_{p_k\to\infty} S_{p_k}. \tag{19}$$

By the properties of the norm $\|\cdot\|_p$ we obtain

$$\delta_* \equiv \|R_*\|_\infty = \lim_{p_k\to\infty} \|R_{p_k}\|_{p_k}, \tag{20}$$

where

$$R_* = \left(r_{ij}^{(*)}\right) = \lim_{p_k\to\infty} R_{p_k}.$$

Let

$$u_{ij}^{(p)} = |r_{ij}^{(p)}|^{p-1}/\|R_p\|_p^{p-1}.$$

Then (see (12))

$$v_{ij}^{(p)} = \operatorname{sign}\left(r_{ij}^{(p)}\right) u_{ij}^{(p)}, \quad 1 < p < \infty. \tag{21}$$

If $0 \leqslant |r_{ij}^{(*)}| < \delta_*$ then

$$0 \leqslant \lim_{p_k \to \infty} \left(u_{ij}^{(p_k)}\right)^{1/(p_k - 1)} = \alpha < 1$$

and consequently

$$\lim_{p_k \to \infty} u_{ij}^{(p_k)} = 0.$$

Therefore (compare Bandler, Charalambous [1])

$$v_{ij}^{(*)} = \lim_{p_k \to \infty} v_{ij}^{(p_k)} = 0 \tag{22}$$

for the indices (i, j) such that $|r_{ij}^{(*)}| < \delta_*$. If $|r_{ij}^{(*)}| = \delta_*$ then

$$\lim_{p_k \to \infty} \left(u_{ij}^{(p_k)}\right)^{1/(p_k - 1)} = 1.$$

This do not imply that there exists

$$\lim_{p_k \to \infty} u_{ij}^{(p_k)}.$$

However, we have

$$0 \leqslant u_{ij}^{(p)} \leqslant 1.$$

For that reason the elements $u_{ij}^{(p_k)}$ are bounded (see (20)) and there exists a convergent subsequence of $\{u_{ij}^{(p_k)}\}$ for (i, j) such that $|r_{ij}^{(*)}| = \delta_*$. Therefore there exists the sequence $\{p_k\}$, $p_k \to \infty$, such that there exist the limit (19) and the limit

$$V_* = \left(v_{ij}^{(*)}\right) = \lim_{p_k \to \infty} V_{p_k}. \tag{23}$$

Theorem 4. *Let the sequence* $\{S_p\}$ *of the stationary points of* (4), $\operatorname{rank}(S_p) = r$, *be bounded and let the sequence* $\{p_k\}$, $p_k \to \infty$, *be such that* (19) *and* (23) *hold. If* $\operatorname{rank}(S_*) = r$ *then* S_* *is a stationary point of* (5) *and* S_* *satisfies condition* RC.

Proof. From (21)–(23) we have

$$\begin{aligned} &v_{ij}^{(*)} \operatorname{sign}\left(r_{ij}^{(*)}\right) \geqslant 0, \quad &(i, j) \in J(R_*), \\ &v_{ij}^{(*)} = 0, \quad &(i, j) \notin J(R_*). \end{aligned} \tag{24}$$

However, from (17) we receive

$$V^T{}_* S_* = 0, \qquad V_* S^T{}_* = 0. \tag{25}$$

The matrix V_* is nonzero because

$$\lim_{p_k \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} v_{ij}^{(p_k)} r_{ij}^{(p_k)} = \lim_{p_k \to \infty} \|R_{p_k}\|_{p_k} = \delta_* \neq 0.$$

Therefore from (24) and (25) the first part of the theorem follows because (15) and (18) hold and $\operatorname{rank}(S_*) = r$.

We omit the index k of the elements of sequence $\{p_k\}$ to make easier notations (see (19), (20), (23)).

Now we prove that the matrix S_* satisfies condition RC. Let $S_p = X_p Y_p$, $S_* = X_* Y_*$, where X_p, X_* are matrices of dimension $m \times r$. Let $y_j^{(p)}$, $y_j^{(*)}$ denote the jth column of matrices Y_p, Y_*, respectively. Suppose that S_* does not satisfy condition RC. Then either there exists t $(1 \leqslant t \leqslant n)$ such that the tth column of Y_* is not a Chebyshev solution of $X_* y = a_t$ or there exists s $(1 \leqslant s \leqslant m)$ such that the sth column of $X^T{}_*$ is not the Chebyshev solution of $Y^T{}_* x = d_s$. We assume the first case. So there exists t such that

$$\| X_* y_t^{(*)} - a_t \|_\infty > \inf_h \| X_* h - a_t \|_\infty .$$

By the argument from proof of Theorem 3 we know that there exist $\varepsilon > 0$ and q_0 such that (see (8), (9))

$$u_t(S_p) > \varepsilon \quad \text{for } p > q_0. \tag{26}$$

Moreover, we have (compare (10), see [8])

$$\beta_t^{(p)} \equiv \| X_p y_t^{(p)} - a_t \|_p = \inf_h \| X_p h - a_t \|_p .$$

Therefore we have (see (26), compare (11))

$$\beta_t^{(p)} > \varepsilon + m^{-1/p} \beta_t^{(p)} \quad \text{for } p > q_0. \tag{27}$$

Let $\beta_t^* = \lim_{p \to \infty} \beta_t^{(p)}$. This limit there exists (see (19)). From (27) we obtain $\beta_t^* \geqslant \varepsilon + \beta_t^*$, which is a contradiction. This completes the proof. □

It is known that each solution of (4) satisfies the conditions (17) and each solution of (5) satisfies the conditions (14) and (15). If additionaly the solution of (5) has rank r then the relations (18) also hold. Thus some properties of the solutions of (4) and (5) are similar to the appropriate properties of the stationary points. We see that under some additional assumptions a cluster point of the sequence of the stationary points of (4) and a cluster point of the sequence of solutions of (1) satisfy condition RC. However, we prove separately these facts because the additional conditions in Theorem 4 are stronger than in Theorem 3.

References

[1] J.W. Bandler and C. Charalambous, On conditions for optimality in least pth approximation with $p \to \infty$, *J. Optim. Theory Appl.* **11** (1973) 556–566.

[2] J. Descloux, Approximations in L^P and Chebyshev approximations, *J. Soc. Indust. Appl. Math.* **11** (1963) 1017–1026.

[3] R. Fletcher, J.A. Grant and M.D. Hebden, Linear minimax approximation as the limit of best l_p-approximation, *SIAM J. Numer. Anal.* **11** (1974) 123–136.

[4] M.R. Osborne and G.A. Watson, Nonlinear approximation problems in vector norms, in: G.A. Watson, Ed., Numerical Analysis (Springer, Berlin, 1978).

[5] J.R. Rice, *The Approximation of Functions, II* (Addison Wesley, London, 1969).

[6] G.A. Watson, *Approximation Theory and Numerical Methods* (Wiley, London, 1980).

[7] K. Ziętak, The properties of the minimax solution of a nonlinear matrix equation $XY = A$, *IMA J. Numer. Anal.* **3** (1983) 229–244.

[8] K. Ziętak, The l_p-solution of the nonlinear matrix equation $XY = A$, *BIT* **23** (1983) 248–257.

[9] K. Ziętak, A note on a characterization of the stationary points of a certain minimax problem, Report Nr N-122, Institute of Computer Science, University of Wrocław, Wrocław 1983.